

29. Laplace Transforms

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29. Laplace Transforms

29.1. Definition of the Laplace Transform

One-dimensional Laplace Transform

$$29.1.1 \quad f(s) = \mathcal{L}\{F(t)\} = \int_0^{\infty} e^{-st} F(t) dt$$

$F(t)$ is a function of the real variable t and s is a complex variable. $F(t)$ is called the original function and $f(s)$ is called the image function. If the integral in 29.1.1 converges for a real $s=s_0$, i.e.,

$$\lim_{\substack{A \rightarrow 0 \\ B \rightarrow \infty}} \int_A^B e^{-s_0 t} F(t) dt$$

exists, then it converges for all s with $\Re s > s_0$, and the image function is a single valued analytic

function of s in the half-plane $\Re s > s_0$.

Two-dimensional Laplace Transform

29.1.2

$$f(u, v) = \mathcal{L}\{F(x, y)\} = \int_0^{\infty} \int_0^{\infty} e^{-ux-vy} F(x, y) dx dy$$

Definition of the Unit Step Function

$$29.1.3 \quad u(t) = \begin{cases} 0 & (t < 0) \\ \frac{1}{2} & (t = 0) \\ 1 & (t > 0) \end{cases}$$

In the following tables the factor $u(t)$ is to be understood as multiplying the original function $F(t)$.

29.2. Operations for the Laplace Transform¹

Original Function $F(t)$

$$29.2.1 \quad F(t)$$

Inversion Formula

$$29.2.2 \quad \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{ts} f(s) ds$$

Linearity Property

$$29.2.3 \quad AF(t) + BG(t)$$

Differentiation

$$29.2.4 \quad F'(t)$$

$$29.2.5 \quad F^{(n)}(t)$$

Integration

$$29.2.6 \quad \int_0^t F(\tau) d\tau$$

$$29.2.7 \quad \int_0^t \int_0^{\tau} F(\lambda) d\lambda d\tau$$

Convolution (Faltung) Theorem

$$29.2.8 \quad \int_0^t F_1(t-\tau) F_2(\tau) d\tau = F_1 * F_2$$

$$29.2.9 \quad -tF(t)$$

$$29.2.10 \quad (-1)^n t^n F(t)$$

Image Function $f(s)$

$$\int_0^{\infty} e^{-st} F(t) dt$$

$$f(s)$$

$$Af(s) + Bg(s)$$

$$sf(s) - F(+0)$$

$$s^n f(s) - s^{n-1} F(+0) - s^{n-2} F'(+0) - \dots - F^{(n-1)}(+0)$$

$$\frac{1}{s} f(s)$$

$$\frac{1}{s^2} f(s)$$

$$f_1(s) f_2(s)$$

Differentiation

$$f'(s)$$

$$f^{(n)}(s)$$

¹ Adapted by permission from R. V. Churchill, Operational mathematics, 2d ed., McGraw-Hill Book Co., Inc., New York, N.Y., 1958.

Original Function $F(t)$		Image Function $f(s)$
		Integration
29.2.11	$\frac{1}{t} F(t)$	$\int_s^\infty f(x) dx$
		Linear Transformation
29.2.12	$e^{at} F(t)$	$f(s-a)$
29.2.13	$\frac{1}{c} F\left(\frac{t}{c}\right) \quad (c>0)$	$f(cs)$
29.2.14	$\frac{1}{c} e^{(b/c)t} F\left(\frac{t}{c}\right) \quad (c>0)$	$f(cs-b)$
		Translation
29.2.15	$F(t-b)u(t-b) \quad (b>0)$	$e^{-bs}f(s)$
		Periodic Functions
29.2.16	$F(t+a)=F(t)$	$\frac{\int_0^a e^{-st} F(t) dt}{1-e^{-as}}$
29.2.17	$F(t+a)=-F(t)$	$\frac{\int_0^a e^{-st} F(t) dt}{1+e^{-as}}$
		Half-Wave Rectification of $F(t)$ in 29.2.17
29.2.18	$F(t) \sum_{n=0}^{\infty} (-1)^n u(t-na)$	$\frac{f(s)}{1-e^{-as}}$
		Full-Wave Rectification of $F(t)$ in 29.2.17
29.2.19	$ F(t) $	$f(s) \coth \frac{as}{2}$
		Heaviside Expansion Theorem
29.2.20	$\sum_{n=1}^m \frac{p(a_n)}{q'(a_n)} e^{a_n t}$	$\frac{p(s)}{q(s)}, q(s)=(s-a_1)(s-a_2) \dots (s-a_m)$
		$p(s)$ a polynomial of degree $< m$
29.2.21	$e^{at} \sum_{n=1}^r \frac{p^{(r-n)}(a)}{(r-n)!} \frac{t^{n-1}}{(n-1)!}$	$\frac{p(s)}{(s-a)^r}$
		$p(s)$ a polynomial of degree $< r$

29.3. Table of Laplace Transforms^{2,3}

For a comprehensive table of Laplace and other integral transforms see [29.9]. For a table of two-dimensional Laplace transforms see [29.11].

	$f(s)$	$F(t)$
29.3.1	$\frac{1}{s}$	1
29.3.2	$\frac{1}{s^2}$	t


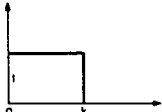
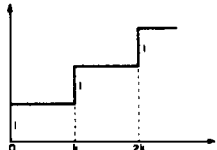
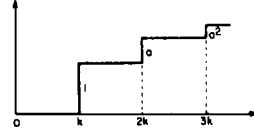
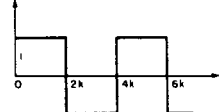
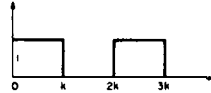
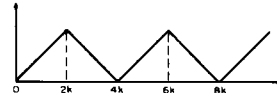
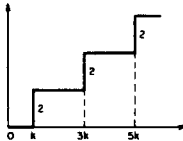
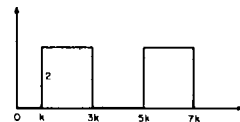
² The numbers in bold type in the $f(s)$ and $F(t)$ columns indicate the chapters in which the properties of the respective higher mathematical functions are given.

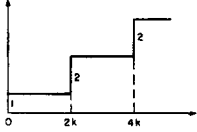
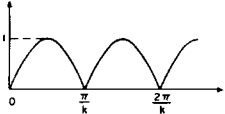

³ Adapted by permission from R. V. Churchill, Operational mathematics, 2d. ed., McGraw-Hill Book Co., Inc., New York, N. Y., 1958.

	$f(s)$		$F(t)$
29.3.3	$\frac{1}{s^n} \quad (n=1, 2, 3, \dots)$		$\frac{t^{n-1}}{(n-1)!}$
29.3.4	$\frac{1}{\sqrt{s}}$		$\frac{1}{\sqrt{\pi t}}$
29.3.5	$s^{-3/2}$		$2\sqrt{t/\pi}$
29.3.6	$s^{-(n+1)} \quad (n=1, 2, 3, \dots)$		$\frac{2^n t^{n-1}}{1 \cdot 3 \cdot 5 \dots (2n-1) \sqrt{\pi}}$
29.3.7	$\frac{\Gamma(k)}{s^k} \quad (k > 0)$	6	t^{k-1}
29.3.8	$\frac{1}{s+a}$		e^{-at}
29.3.9	$\frac{1}{(s+a)^2}$		$t e^{-at}$
29.3.10	$\frac{1}{(s+a)^n} \quad (n=1, 2, 3, \dots)$		$\frac{t^{n-1} e^{-at}}{(n-1)!}$
29.3.11	$\frac{\Gamma(k)}{(s+a)^k} \quad (k > 0)$	6	$t^{k-1} e^{-at}$
29.3.12	$\frac{1}{(s+a)(s+b)} \quad (a \neq b)$		$\frac{e^{-at} - e^{-bt}}{b-a}$
29.3.13	$\frac{s}{(s+a)(s+b)} \quad (a \neq b)$		$\frac{a e^{-at} - b e^{-bt}}{a-b}$
29.3.14	$\frac{1}{(s+a)(s+b)(s+c)}$ (a, b, c distinct constants)		$-\frac{(b-c)e^{-at} + (c-a)e^{-bt} + (a-b)e^{-ct}}{(a-b)(b-c)(c-a)}$
29.3.15	$\frac{1}{s^2+a^2}$		$\frac{1}{a} \sin at$
29.3.16	$\frac{s}{s^2+a^2}$		$\cos at$
29.3.17	$\frac{1}{s^2-a^2}$		$\frac{1}{a} \sinh at$
29.3.18	$\frac{s}{s^2-a^2}$		$\cosh at$
29.3.19	$\frac{1}{s(s^2+a^2)}$		$\frac{1}{a^2} (1 - \cos at)$
29.3.20	$\frac{1}{s^2(s^2+a^2)}$		$\frac{1}{a^3} (at - \sin at)$
29.3.21	$\frac{1}{(s^2+a^2)^2}$		$\frac{1}{2a^3} (\sin at - at \cos at)$

	$f(s)$	$F(t)$	
29.3.22	$\frac{s}{(s^2+a^2)^2}$	$\frac{t}{2a} \sin at$	
29.3.23	$\frac{s^2}{(s^2+a^2)^2}$	$\frac{1}{2a} (\sin at + at \cos at)$	
29.3.24	$\frac{s^2-a^2}{(s^2+a^2)^2}$	$t \cos at$	
29.3.25	$\frac{s}{(s^2+a^2)(s^2+b^2)} \quad (a^2 \neq b^2)$	$\frac{\cos at - \cos bt}{b^2 - a^2}$	
29.3.26	$\frac{1}{(s+a)^2 + b^2}$	$\frac{1}{b} e^{-at} \sin bt$	
29.3.27	$\frac{s+a}{(s+a)^2 + b^2}$	$e^{-at} \cos bt$	
29.3.28	$\frac{3a^2}{s^3 + a^3}$	$e^{-at} - e^{iat} \left(\cos \frac{at\sqrt{3}}{2} - \sqrt{3} \sin \frac{at\sqrt{3}}{2} \right)$	
29.3.29	$\frac{4a^3}{s^4 + 4a^4}$	$\sin at \cosh at - \cos at \sinh at$	
29.3.30	$\frac{s}{s^4 + 4a^4}$	$\frac{1}{2a^2} \sin at \sinh at$	
29.3.31	$\frac{1}{s^4 - a^4}$	$\frac{1}{2a^3} (\sinh at - \sin at)$	
29.3.32	$\frac{s}{s^4 - a^4}$	$\frac{1}{2a^2} (\cosh at - \cos at)$	
29.3.33	$\frac{8a^3 s^2}{(s^2+a^2)^3}$	$(1+a^2 t^2) \sin at - at \cos at$	
29.3.34	$\frac{1}{s} \left(\frac{s-1}{s} \right)^n$	$L_n(t)$	22
29.3.35	$\frac{s}{(s+a)^{\frac{1}{2}}}$	$\frac{1}{\sqrt{\pi t}} e^{-at} (1-2at)$	
29.3.36	$\sqrt{s+a} - \sqrt{s+b}$	$\frac{1}{2\sqrt{\pi t^3}} (e^{-bt} - e^{-at})$	
29.3.37	$\frac{1}{\sqrt{s+a}}$	$\frac{1}{\sqrt{\pi t}} - ae^{at} \operatorname{erfc} a\sqrt{t}$	7
29.3.38	$\frac{\sqrt{s}}{s-a^2}$	$\frac{1}{\sqrt{\pi t}} + ae^{at} \operatorname{erf} a\sqrt{t}$	7
29.3.39	$\frac{\sqrt{s}}{s+a^2}$	$\frac{1}{\sqrt{\pi t}} - \frac{2a}{\sqrt{\pi}} e^{-a^2 t} \int_0^{a\sqrt{t}} e^{\lambda^2} d\lambda$	7
29.3.40	$\frac{1}{\sqrt{s}(s-a^2)}$	$\frac{1}{a} e^{at} \operatorname{erf} a\sqrt{t}$	7

	$f(s)$	$F(t)$	
29.3.41	$\frac{1}{\sqrt{s}(s+a^2)}$	$\frac{2}{a\sqrt{\pi}} e^{-a^2 t} \int_0^{a\sqrt{t}} e^{\lambda^2} d\lambda$	7
29.3.42	$\frac{b^2-a^2}{(s-a^2)(b+\sqrt{s})}$	$e^{a^2 t} [b-a \operatorname{erf} a\sqrt{t}] - b e^{b^2 t} \operatorname{erfc} b\sqrt{t}$	7
29.3.43	$\frac{1}{\sqrt{s}(\sqrt{s}+a)}$	$e^{a^2 t} \operatorname{erfc} a\sqrt{t}$	7
29.3.44	$\frac{1}{(s+a)\sqrt{s+b}}$	$\frac{1}{\sqrt{b-a}} e^{-at} \operatorname{erf} (\sqrt{b-a}\sqrt{t})$	7
29.3.45	$\frac{b^2-a^2}{\sqrt{s}(s-a^2)(\sqrt{s}+b)}$	$e^{a^2 t} \left[\frac{b}{a} \operatorname{erf} (a\sqrt{t}) - 1 \right] + e^{b^2 t} \operatorname{erfc} b\sqrt{t}$	7
29.3.46	$\frac{(1-s)^n}{s^{n+\frac{1}{2}}}$	$\frac{n!}{(2n)!\sqrt{\pi t}} H_{2n}(\sqrt{t})$	22
29.3.47	$\frac{(1-s)^n}{s^{n+\frac{1}{2}}}$	$\frac{n!}{(2n+1)!\sqrt{\pi}} H_{2n+1}(\sqrt{t})$	22
29.3.48	$\frac{\sqrt{s+2a}-1}{\sqrt{s}}$	$ae^{-at} [I_1(at) + I_0(at)]$	9
29.3.49	$\frac{1}{\sqrt{s+a}\sqrt{s+b}}$	$e^{-\frac{1}{2}(a+b)t} I_0\left(\frac{a-b}{2}t\right)$	9
29.3.50	$\frac{\Gamma(k)}{(s+a)^k(s+b)^k} \quad (k>0) \quad 6$	$\sqrt{\pi} \left(\frac{t}{a-b}\right)^{k-\frac{1}{2}} e^{-\frac{1}{2}(a+b)t} I_{k-\frac{1}{2}}\left(\frac{a-b}{2}t\right)$	10
29.3.51	$\frac{1}{(s+a)^{\frac{1}{2}}(s+b)^{\frac{1}{2}}}$	$te^{-\frac{1}{2}(a+b)t} \left[I_0\left(\frac{a-b}{2}t\right) + I_1\left(\frac{a-b}{2}t\right) \right]$	9
29.3.52	$\frac{\sqrt{s+2a}-\sqrt{s}}{\sqrt{s+2a}+\sqrt{s}}$	$\frac{1}{t} e^{-at} I_1(at)$	9
29.3.53	$\frac{(a-b)^k}{(\sqrt{s+a}+\sqrt{s+b})^{2k}} \quad (k>0)$	$\frac{k}{t} e^{-\frac{1}{2}(a+b)t} I_k\left(\frac{a-b}{2}t\right)$	9
29.3.54	$\frac{(\sqrt{s+a}+\sqrt{s})^{-2\nu}}{\sqrt{s}\sqrt{s+a}} \quad (\nu>-1)$	$\frac{1}{a^\nu} e^{-\frac{1}{2}at} I_\nu\left(\frac{1}{2}at\right)$	9
29.3.55	$\frac{1}{\sqrt{s^2+a^2}}$	$J_0(at)$	9
29.3.56	$\frac{(\sqrt{s^2+a^2}-s)^\nu}{\sqrt{s^2+a^2}} \quad (\nu>-1)$	$a^\nu J_\nu(at)$	9
29.3.57	$\frac{1}{(s^2+a^2)^k} \quad (k>0)$	$\frac{\sqrt{\pi}}{\Gamma(k)} \left(\frac{t}{2a}\right)^{k-\frac{1}{2}} J_{k-\frac{1}{2}}(at)$	6, 10

$f(s)$	$F(t)$	
29.3.58 $(\sqrt{s^2+a^2}-s)^k \quad (k>0)$	$\frac{ka^k}{t} J_k(at)$	9
29.3.59 $\frac{(s-\sqrt{s^2-a^2})^\nu}{\sqrt{s^2-a^2}} \quad (\nu>-1)$	$a^\nu I_\nu(at)$	9
29.3.60 $\frac{1}{(s^2-a^2)^k} \quad (k>0)$	$\frac{\sqrt{\pi}}{\Gamma(k)} \left(\frac{t}{2a}\right)^{k-1} I_{k-1}(at)$	6, 10
29.3.61 $\frac{1}{s} e^{-ks}$	$u(t-k)$	
29.3.62 $\frac{1}{s^2} e^{-ks}$	$(t-k)u(t-k)$	
29.3.63 $\frac{1}{s^\mu} e^{-ks} \quad (\mu>0)$	$\frac{(t-k)^{\mu-1}}{\Gamma(\mu)} u(t-k)$	6
29.3.64 $\frac{1-e^{-ks}}{s}$	$u(t)-u(t-k)$	
29.3.65 $\frac{1}{s(1-e^{-ks})} = \frac{1+\coth \frac{1}{2}ks}{2s}$	$\sum_{n=0}^{\infty} u(t-nk)$	
29.3.66 $\frac{1}{s(e^{ks}-a)}$	$\sum_{n=1}^{\infty} a^{n-1} u(t-nk)$	
29.3.67 $\frac{1}{s} \tanh ks$	$u(t)+2 \sum_{n=1}^{\infty} (-1)^n u(t-2nk)$	
29.3.68 $\frac{1}{s(1+e^{-ks})}$	$\sum_{n=0}^{\infty} (-1)^n u(t-nk)$	
29.3.69 $\frac{1}{s^2} \tanh ks$	$tu(t)+2 \sum_{n=1}^{\infty} (-1)^n (t-2nk)u(t-2nk)$	
29.3.70 $\frac{1}{s \sinh ks}$	$2 \sum_{n=0}^{\infty} u[t-(2n+1)k]$	
29.3.71 $\frac{1}{s \cosh ks}$	$2 \sum_{n=0}^{\infty} (-1)^n u[t-(2n+1)k]$	

$f(s)$	$F(t)$
29.3.72 $\frac{1}{s} \coth ks$	$u(t) + 2 \sum_{n=1}^{\infty} u(t-2nk)$ 
29.3.73 $\frac{k}{s^2+k^2} \coth \frac{\pi s}{2k}$	$ \sin kt $ 
29.3.74 $\frac{1}{(s^2+1)(1-e^{-\pi s})}$	$\sum_{n=0}^{\infty} (-1)^n u(t-n\pi) \sin t$ 
29.3.75 $\frac{1}{s} e^{-\frac{k}{s}}$	$J_0(2\sqrt{kt})$ 9
29.3.76 $\frac{1}{\sqrt{s}} e^{-\frac{k}{s}}$	$\frac{1}{\sqrt{\pi t}} \cos 2\sqrt{kt}$
29.3.77 $\frac{1}{\sqrt{s}} e^{\frac{k}{s}}$	$\frac{1}{\sqrt{\pi t}} \cosh 2\sqrt{kt}$
29.3.78 $\frac{1}{s^{3/2}} e^{-\frac{k}{s}}$	$\frac{1}{\sqrt{\pi k}} \sin 2\sqrt{kt}$
29.3.79 $\frac{1}{s^{3/2}} e^{\frac{k}{s}}$	$\frac{1}{\sqrt{\pi k}} \sinh 2\sqrt{kt}$
29.3.80 $\frac{1}{s^\mu} e^{-\frac{k}{s}} \quad (\mu > 0)$	$\left(\frac{t}{k}\right)^{\frac{\mu-1}{2}} J_{\mu-1}(2\sqrt{kt})$ 9
29.3.81 $\frac{1}{s^\mu} e^{\frac{k}{s}} \quad (\mu > 0)$	$\left(\frac{t}{k}\right)^{\frac{\mu-1}{2}} I_{\mu-1}(2\sqrt{kt})$ 9
29.3.82 $e^{-k\sqrt{s}} \quad (k > 0)$	$\frac{k}{2\sqrt{\pi t^3}} \exp\left(-\frac{k^2}{4t}\right)$
29.3.83 $\frac{1}{s} e^{-k\sqrt{s}} \quad (k \geq 0)$	$\operatorname{erfc} \frac{k}{2\sqrt{t}}$ 7
29.3.84 $\frac{1}{\sqrt{s}} e^{-k\sqrt{s}} \quad (k \geq 0)$	$\frac{1}{\sqrt{\pi t}} \exp\left(-\frac{k^2}{4t}\right)$
29.3.85 $\frac{1}{s^{\frac{3}{2}}} e^{-k\sqrt{s}} \quad (k \geq 0)$	$2\sqrt{\frac{t}{\pi}} \exp\left(-\frac{k^2}{4t}\right) - k \operatorname{erfc} \frac{k}{2\sqrt{t}} = 2\sqrt{t} \operatorname{erfc} \frac{k}{2\sqrt{t}}$ 7
29.3.86 $\frac{1}{s^{1+i^n}} e^{-k\sqrt{s}} \quad (n=0, 1, 2, \dots; k \geq 0)$	$(4t)^{i^n} \operatorname{erfc} \frac{k}{2\sqrt{t}}$ 7
29.3.87 $\frac{s^{-n-1}}{s^{\frac{n-1}{2}}} e^{-k\sqrt{s}} \quad (n=0, 1, 2, \dots; k > 0)$	$\frac{\exp\left(-\frac{k^2}{4t}\right)}{2^n \sqrt{\pi t^{n+1}}} H_n\left(\frac{k}{2\sqrt{t}}\right)$ 22
29.3.88 $\frac{e^{-k\sqrt{s}}}{a+\sqrt{s}} \quad (k \geq 0)$	$\frac{1}{\sqrt{\pi t}} \exp\left(-\frac{k^2}{4t}\right) - ae^{ak^2} \operatorname{erfc}\left(a\sqrt{t} + \frac{k}{2\sqrt{t}}\right)$ 7

	$f(s)$		$F(t)$	
29.3.89	$\frac{ae^{-k\sqrt{s}}}{s(a+\sqrt{s})} \quad (k \geq 0)$		$-e^{ak}e^{a^2t} \operatorname{erfc}\left(a\sqrt{t} + \frac{k}{2\sqrt{t}}\right) + \operatorname{erfc} \frac{k}{2\sqrt{t}}$	7
29.3.90	$\frac{e^{-k\sqrt{s}}}{\sqrt{s}(a+\sqrt{s})} \quad (k \geq 0)$		$e^{ak}e^{a^2t} \operatorname{erfc}\left(a\sqrt{t} + \frac{k}{2\sqrt{t}}\right)$	7
29.3.91	$\frac{e^{-k\sqrt{s(s+a)}}}{\sqrt{s(s+a)}} \quad (k \geq 0)$		$e^{-\frac{1}{2}at} I_0\left(\frac{1}{2}a\sqrt{t^2-k^2}\right) u(t-k)$	9
29.3.92	$\frac{e^{-k\sqrt{s^2+a^2}}}{\sqrt{s^2+a^2}} \quad (k \geq 0)$		$J_0(a\sqrt{t^2-k^2}) u(t-k)$	9
29.3.93	$\frac{e^{-k\sqrt{s^2-a^2}}}{\sqrt{s^2-a^2}} \quad (k \geq 0)$		$I_0(a\sqrt{t^2-k^2}) u(t-k)$	9
29.3.94	$\frac{e^{-k(\sqrt{s^2+a^2}-s)}}{\sqrt{s^2+a^2}} \quad (k \geq 0)$		$J_0(a\sqrt{t^2+2kt})$	9
29.3.95	$e^{-ks} - e^{-k\sqrt{s^2+a^2}} \quad (k > 0)$		$\frac{ak}{\sqrt{t^2-k^2}} J_1(a\sqrt{t^2-k^2}) u(t-k)$	9
29.3.96	$e^{-k\sqrt{s^2-a^2}} - e^{-ks} \quad (k > 0)$		$\frac{ak}{\sqrt{t^2-k^2}} I_1(a\sqrt{t^2-k^2}) u(t-k)$	9
29.3.97	$\frac{a^\nu e^{-k\sqrt{s^2+a^2}}}{\sqrt{s^2+a^2}(\sqrt{s^2+a^2}+s)^\nu} \quad (\nu > -1, k \geq 0)$		$\left(\frac{t-k}{t+k}\right)^{\frac{1}{2}\nu} J_\nu(a\sqrt{t^2-k^2}) u(t-k)$	9
29.3.98	$\frac{1}{s} \ln s$		$-\gamma - \ln t \quad (\gamma = .57721\ 56649 \dots \text{Euler's constant})$	
29.3.99	$\frac{1}{s^k} \ln s \quad (k > 0)$		$\frac{t^{k-1}}{\Gamma(k)} [\psi(k) - \ln t]$	6
29.3.100	$\frac{\ln s}{s-a} \quad (a > 0)$		$e^{at} [\ln a + E_1(at)]$	5
29.3.101	$\frac{\ln s}{s^2+1}$		$\cos t \operatorname{Si}(t) - \sin t \operatorname{Ci}(t)$	5
29.3.102	$\frac{s \ln s}{s^2+1}$		$-\sin t \operatorname{Si}(t) - \cos t \operatorname{Ci}(t)$	5
29.3.103	$\frac{1}{s} \ln(1+ks) \quad (k > 0)$		$E_1\left(\frac{t}{k}\right)$	5
29.3.104	$\ln \frac{s+a}{s+b}$		$\frac{1}{t} (e^{-bt} - e^{-at})$	
29.3.105	$\frac{1}{s} \ln(1+k^2s^2) \quad (k > 0)$		$-2 \operatorname{Ci}\left(\frac{t}{k}\right)$	5
29.3.106	$\frac{1}{s} \ln(s^2+a^2) \quad (a > 0)$		$2 \ln a - 2 \operatorname{Ci}(at)$	5

	$f(s)$		$F(t)$	
29.3.107	$\frac{1}{s^2} \ln(s^2 + a^2) \quad (a > 0)$		$\frac{2}{a} [at \ln a + \sin at - at \operatorname{Ci}(at)]$	5
29.3.108	$\ln \frac{s^2 + a^2}{s^2}$		$\frac{2}{t} (1 - \cos at)$	
29.3.109	$\ln \frac{s^2 - a^2}{s^2}$		$\frac{2}{t} (1 - \cosh at)$	
29.3.110	$\arctan \frac{k}{s}$		$\frac{1}{t} \sin kt$	
29.3.111	$\frac{1}{s} \arctan \frac{k}{s}$		$\operatorname{Si}(kt)$	5
29.3.112	$e^{ks^2} \operatorname{erfc} ks \quad (k > 0)$	7	$\frac{1}{k\sqrt{\pi}} \exp\left(-\frac{t^2}{4k^2}\right)$	
29.3.113	$\frac{1}{s} e^{ks^2} \operatorname{erfc} ks \quad (k > 0)$	7	$\operatorname{erf} \frac{t}{2k}$	7
29.3.114	$e^{ks} \operatorname{erfc} \sqrt{ks} \quad (k > 0)$	7	$\frac{\sqrt{k}}{\pi \sqrt{t}(t+k)}$	
29.3.115	$\frac{1}{\sqrt{s}} \operatorname{erfc} \sqrt{ks} \quad (k \geq 0)$	7	$\frac{1}{\sqrt{\pi t}} u(t-k)$	
29.3.116	$\frac{1}{\sqrt{s}} e^{ks} \operatorname{erfc} \sqrt{ks} \quad (k \geq 0)$	7	$\frac{1}{\sqrt{\pi(t+k)}}$	
29.3.117	$\operatorname{erf} \frac{k}{\sqrt{s}}$	7	$\frac{1}{\pi t} \sin 2k\sqrt{t}$	
29.3.118	$\frac{1}{\sqrt{s}} e^{\frac{k^2}{s}} \operatorname{erfc} \frac{k}{\sqrt{s}}$	7	$\frac{1}{\sqrt{\pi t}} e^{-2k\sqrt{t}}$	
29.3.119	$K_0(ks) \quad (k > 0)$	9	$\frac{1}{\sqrt{t^2 - k^2}} u(t-k)$	
29.3.120	$K_0(k\sqrt{s}) \quad (k > 0)$	9	$\frac{1}{2t} \exp\left(-\frac{k^2}{4t}\right)$	
29.3.121	$\frac{1}{s} e^{ks} K_1(ks) \quad (k > 0)$	9	$\frac{1}{k} \sqrt{t(t+2k)}$	
29.3.122	$\frac{1}{\sqrt{s}} K_1(k\sqrt{s}) \quad (k > 0)$	9	$\frac{1}{k} \exp\left(-\frac{k^2}{4t}\right)$	
29.3.123	$\frac{1}{\sqrt{s}} e^{\frac{k}{s}} K_0\left(\frac{k}{s}\right) \quad (k > 0)$	9	$\frac{2}{\sqrt{\pi t}} K_0(2\sqrt{2kt})$	9
29.3.124	$\pi e^{-ks} I_0(ks) \quad (k > 0)$	9	$\frac{1}{\sqrt{t(2k-t)}} [u(t) - u(t-2k)]$	
29.3.125	$e^{-ks} I_1(ks) \quad (k > 0)$	9	$\frac{k-t}{\pi k \sqrt{t(2k-t)}} [u(t) - u(t-2k)]$	

	$f(s)$		$F(t)$
29.3.126	$e^{as}E_1(as) \quad (a>0)$	5	$\frac{1}{t+a}$
29.3.127	$\frac{1}{a}-se^{as}E_1(as) \quad (a>0)$	5	$\frac{1}{(t+a)^2}$
29.3.128	$a^{1-n}e^{as}E_n(as) \quad (a>0; n=0, 1, 2, \dots)$	5	$\frac{1}{(t+a)^n}$
29.3.129	$\left[\frac{\pi}{2}-\text{Si}(s)\right]\cos s+\text{Ci}(s)\sin s$	5	$\frac{1}{t^2+1}$

29.4. Table of Laplace-Stieltjes Transforms ⁴

	$\phi(s)$	$\Phi(t)$
29.4.1	$\int_0^\infty e^{-st}d\Phi(t)$	$\Phi(t)$
29.4.2	$e^{-ks} \quad (k>0)$	$u(t-k)$
29.4.3	$\frac{1}{1-e^{-ks}} \quad (k>0)$	$\sum_{n=0}^\infty u(t-nk)$
29.4.4	$\frac{1}{1+e^{-ks}} \quad (k>0)$	$\sum_{n=0}^\infty (-1)^n u(t-nk)$
29.4.5	$\frac{1}{\sinh ks} \quad (k>0)$	$2 \sum_{n=0}^\infty u[t-(2n+1)k]$
29.4.6	$\frac{1}{\cosh ks} \quad (k>0)$	$2 \sum_{n=0}^\infty (-1)^n u[t-(2n+1)k]$
29.4.7	$\tanh ks \quad (k>0)$	$u(t)+2 \sum_{n=1}^\infty (-1)^n u(t-2nk)$
29.4.8	$\frac{1}{\sinh (ks+a)} \quad (k>0)$	$2 \sum_{n=0}^\infty e^{-(2n+1)a} u[t-(2n+1)k]$
29.4.9	$\frac{e^{-hs}}{\sinh (ks+a)} \quad (k>0, h>0)$	$2 \sum_{n=0}^\infty e^{-(2n+1)a} u[t-h-(2n+1)k]$
29.4.10	$\frac{\sinh (hs+b)}{\sinh (ks+a)} \quad (0<h<k)$	$\sum_{n=0}^\infty e^{-(2n+1)a} \{ e^b u[t+h-(2n+1)k] - e^{-b} u[t-h-(2n+1)k] \}$
29.4.11	$\sum_{n=0}^\infty a_n e^{-k_n s} \quad (0<k_0<k_1<\dots)$	$\sum_{n=0}^\infty a_n u(t-k_n)$

For the definition of the Laplace-Stieltjes transform see [29.7]. In practice, Laplace-Stieltjes transforms are often written as ordinary Laplace transforms involving Dirac's delta function $\delta(t)$. This "function" may formally be considered as

the derivative of the unit step function, $du(t)=\delta(t)dt$, so that $\int_{-\infty}^x du(t)=\int_{-\infty}^x \delta(t)dt=\begin{cases} 0 & (x<0) \\ 1 & (x>0) \end{cases}$. The correspondence 29.4.2, for instance, then assumes the form $e^{-ks}=\int_0^\infty e^{-st}\delta(t-k)dt$.

⁴ Adapted by permission from P. M. Morse and H. Feshbach, *Methods of theoretical physics*, vols. 1, 2, McGraw-Hill Book Co., Inc., New York, N.Y., 1953.

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